## ON A DIFFERENTIAL INEQUALITY OF CESARI AND TURNER

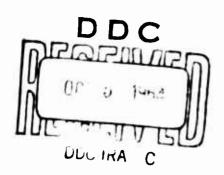
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### SUMMARY

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### 1. Introduction

In a recent paper, [1], Cesari and Turner give a proof of the following

Lemma. If f(x),  $0 \le x \le a$ , is any nonnegative  $L^n$ -integrable function with

(1) 
$$kf^{m}(x) \geq (a - x)^{p} \int_{x}^{a} (a - u)^{q} f^{n}(u) du > 0,$$

for almost all x in [0,a], and, in particular, for x = 0, where k, m, n, p, q > 0 are given constants with m > n, then

(2) 
$$a \leq cf^{r}(0)$$
,

where c and r are constants depending only on k, m, n, p, and q.

The authors give a very ingenious proof of this result, which is not quite direct. Here we shall show that some preliminary transformations enable us to present a simple, direct proof, and indicate some generalizations.

### 2. Proof of Lemma

Let us begin with the transformation

(1) 
$$f(x) = (a - x)^{p/m} w(x),$$

which converts (1.1) into an inequality of the form

(2) 
$$k w(x)^{m} \ge \int_{x}^{a} (a - u)w^{n}(u)du.$$

Now replace  $w^n(u)$  by z(u), and then x by a - x. If we set y(x) = z(a - x) for  $0 \le x \le a$ , we obtain a new inequality of the form

(3) 
$$k y(x)^{c} \geq \int_{0}^{1} y(u)u^{b}du,$$

where c > 1.

The change of independent variable  $x^{b+1} = r$ , and the further change of dependent variable  $y(x^{1/(b+1)}) = g(x)$ , yields the simplified inequality

(4) 
$$k_1 g(x)^c \ge o_0^{/x} g(r) dr,$$

valid for almost all x in [0,a], and valid for x = a, with c > 1 and  $g(x) \ge 0$ .

From this we wish to conclude that

(5) 
$$k_2 g(a)^{c_1} \ge a$$
,

where  $c_1$  depends on c, and  $k_2$  on  $k_1$ . This is easily done. Let  $h(x) = \int_0^{x} g(r) dr$ . Then (4)

is equivalent to

(6) 
$$h^{-1/c} \frac{dh}{dx} \ge k_3.$$

Integrating both sides between 0 and x, we obtain

(7) 
$$h^{(1-1/c)}(x) \ge k_3 x$$
,

since h(0) = 0. Setting x = a, we have

(8) 
$$h(a) \ge k_{ij} a^{c/(c-1)}$$
.

Combining (4) and (8), we have

(9) 
$$k_1 g(a)^c \ge h(a) \ge k_4 a^{c/(n-1)}$$
,

which yields the desired result.

#### 3. Discussion

The same techniques can be used to obtain a corresponding inequality for the case where we start with a relation

(1) 
$$F(f(x)) \geq G(a-x) \int_{x}^{a} H(a-u)K(f(u))du,$$

under appropriate assumptions concerning the functions F, G, H, and K.

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#### REFERENCES

 Cesari, L., and L. H. Turner, "On a Lemma in the Direct Method of the Calculus of Variations," <u>Rendiconti del</u> <u>Circolo Matematico di Palermo</u>, Serie II-Tomo VI, 1957, pp. 109-113.